Torus quotient of Richardson varieties

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Outline

Notations

1. We work over $\mathbb{C}$. Fix $n, r \in \mathbb{Z}_{>0}$.

2. $G = SL(n)$.

3. $T \subseteq G$ is group of diagonal matrices (a maximal torus of G).

4. $B, B^- \subseteq G$ be Borel and opposite Borel subgroups of G containing T.

   These are just upper triangular and lower triangular matrices in G.

5. $P \subseteq G$ is a subgroup containing all block matrices of the form $\begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix}$

   where $a_{11} \in GL(r)$.

Grassmannian as a set

$$G_{r,n} = \Big\{ W \subseteq \mathbb{C}^n \mid \mathrm{dim}(W) = r \Big\}.$$

Any full rank matrix in $\mathbb{C}^{n\times r}$ represents a $W \in G_{r,n}$.

Two full rank matrices $a, b \in \mathbb{C}^{n\times r}$ represent same subspace iff $\exists g \in GL(r)$ such that $a = bg$.

Grassmannian as a set

Two full rank matrices $a, b \in \mathbb{C}^{n\times r}$ represent same subspace iff $\exists g \in GL(r)$ such that $a = bg$.

In other words, orbits of $GL(r)$ action from right on full rank matrices in $\mathbb{C}^{n\times r}$ is $G_{r,n}$. Hence

$$G_{r,n} = (\mathbb{C}^{n\times r})^o / GL(r) = \Big\{ a \cdot GL_r \mid a \in (\mathbb{C}^{n\times r})^o \Big\}.$$

Grassmannian as a set

$$G_{r,n} = (\mathbb{C}^{n\times r})^o / GL(r) = \Big\{ a \cdot GL_r \mid a \in (\mathbb{C}^{n\times r})^o \Big\}.$$

Consider the set

$$G/P = \Big\{ g \cdot P \mid g \in G \Big\}$$

Observe that there is bijection between $G_{r,n}$ and G/P.

Grassmannian variety

Theorem

G/P is a projective variety.

Bruhat Decomposition in type A

$$G = \bigsqcup_{w \in S_n} BwB$$

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Splitting of element g of G as bwb' follows from Gaussian elimination.

Disjointness follows from: BwB are double cosets $B\backslash G/B$.

Bruhat Decomposition in type A

$$G = \bigsqcup_{w \in S_n} BwB$$

Modify to

$$G = \bigsqcup_{w \in ?} BwP$$

How to achieve this? Observe which cosets are merged by P.

Bruhat Decomposition in type A

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How to achieve this? Observe which cosets are merged by P.

Observe permutations $w = \begin{pmatrix} w_{11} & 0 \\ 0 & w_{22} \end{pmatrix}$ are in P.

Notation: $W_r = \{w \in S_n | w \in S_r \times S_{n-r}\}$

Example

$n = 4, r = 2$

$$\begin{aligned} SL(4) = {} & (1,2,3,4) \sqcup (1,2,4,3) \sqcup (2,1,3,4) \sqcup (2,1,4,3) \\ & \sqcup (1,3,2,4) \sqcup (1,3,4,2) \sqcup (3,1,2,4) \sqcup (3,1,4,2) \\ & \sqcup (1,4,2,3) \sqcup (1,4,3,2) \sqcup (4,1,2,3) \sqcup (4,1,3,2) \\ & \sqcup (2,3,1,4) \sqcup (2,3,4,1) \sqcup (3,2,1,4) \sqcup (3,2,4,1) \\ & \sqcup (2,4,1,3) \sqcup (2,4,3,1) \sqcup (4,2,1,3) \sqcup (4,2,3,1) \\ & \sqcup (3,4,1,2) \sqcup (3,4,2,1) \sqcup (4,3,1,2) \sqcup (4,3,2,1) \end{aligned}$$

Bruhat Decomposition in type A

In general we need coset representatives of S_n/W_r. Hence

$$G = \bigsqcup_{wW_r \in S_n/W_r} BwP$$

Schubert varieties

$$G/P = \bigsqcup_{wW_r \in S_n/W_r} BwP/P = \bigsqcup_{wW_r \in S_n/W_r} BwP$$

BwP are called Schubert cells and their closures are Schubert varieties.

Notation: $X_w = \overline{BwB}$

Richardson Variety

There is another way to decompose G,

$$G/P = \bigsqcup_{vW_r \in S_n/W_r} B^- vP$$

Here $B^- vP$ are called opposite Schubert cells and their closures are opposite Schubert varieties.

Notation: $X^v = \overline{BvB}$

Richardson Variety

Fix cosets vW_r, wW_r, The Richardson variety X_v^w is defined as

$$X_w^v = \overline{B^- vP} \cap \overline{BwP}$$

Coset representatives

How to chose?

Set of coset representatives is

$$\Big\{ w \in S_n \mid w(1) < \cdots < w(r), w(r+1) < \cdots < w(n) \Big\}.$$

Observe that $w(1), \ldots, w(r)$ is sufficient to describe a coset. Hence

$$I_{r,n} = \Big\{ \underline{i} = (i_1, \ldots, i_r) \mid 1 \leq i_1 < \ldots < i_r \leq n \Big\}$$

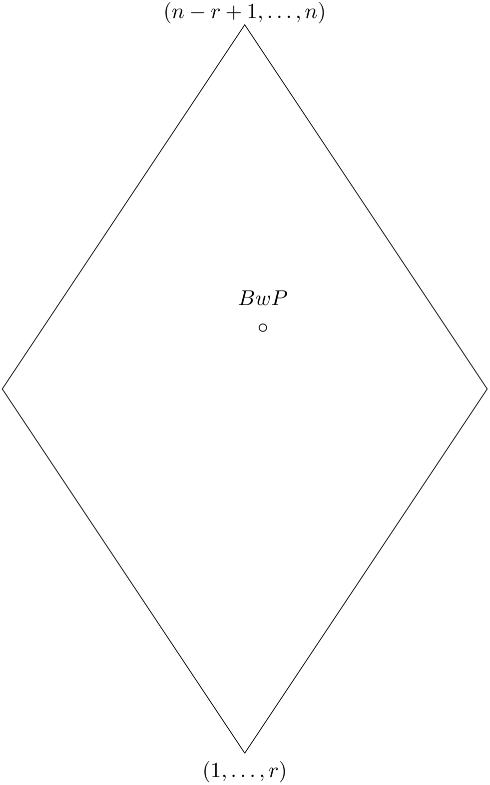
is set of coset representatives.

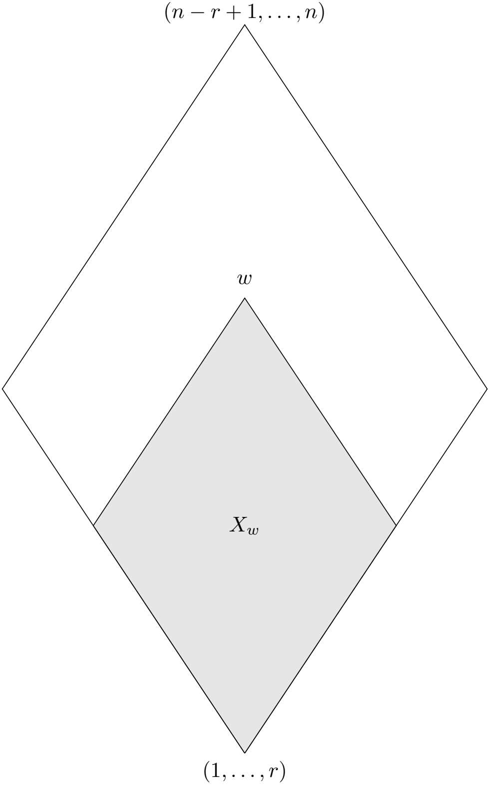
Bruhat Order

On the set $I_{r,n}$

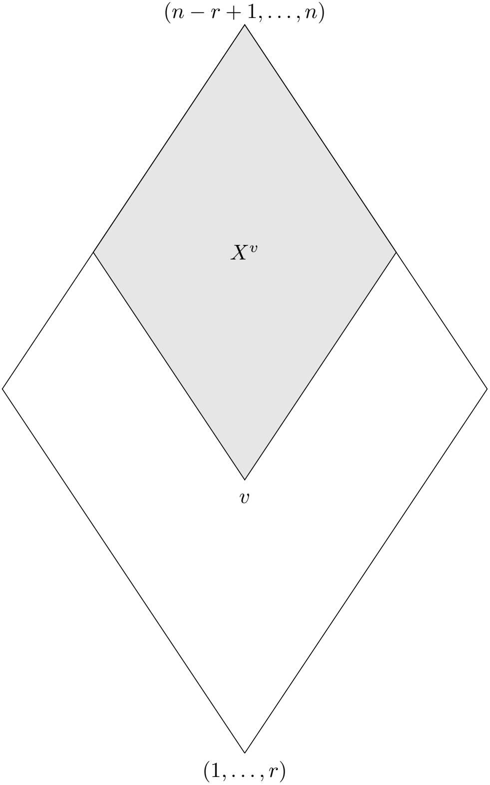
We define Bruhat order on $I_{r,n}$ as follows

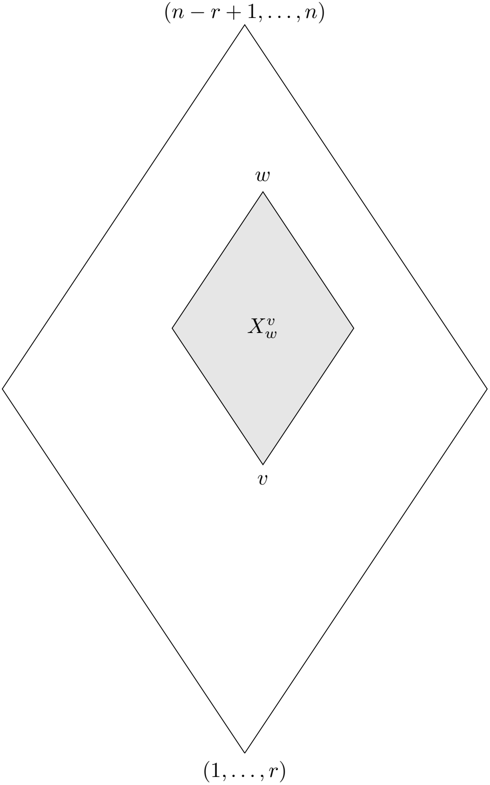
$$\underline{i} \leq \underline{j} \text{ if } i_1 \leq j_1, \ldots, i_r \leq j_r$$


$(n-r+1,\ldots,n)$
BwP
$(1,\ldots,r)$



$(n-r+1,\ldots,n)$
w
X_w
$(1,\ldots,r)$


$(n-r+1,\ldots,n)$
X^v
v
$(1,\ldots,r)$


$(n-r+1,\ldots,n)$
w
X_w^v
v
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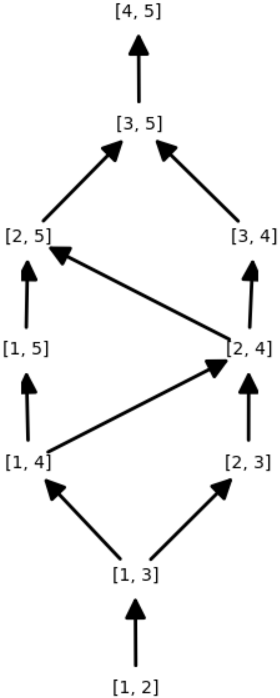
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[3, 5]
[2, 5]
[3, 4]
[1, 5]
[2, 4]
[1, 4]
[2, 3]
[1, 3]
[1, 2]

Group action on variety

Let X be a projective variety with embedding, $\mathbb{C}[X]$ be the homogeneous coordinate ring.

1. Action is a map $G \times X \to X$ of varieties.
2. Induced action on $\mathbb{C}[X]$ is a map $G \times \mathbb{C}[X] \to \mathbb{C}[X]$ defined by $g \cdot f(x) = f(g^{-1} \cdot x)$.
3. Ring of invariants: $\mathbb{C}[X]^G = \left\{ f \in \mathbb{C}[X] \mid \forall g \in G, g \cdot f = f \right\}$

Projective GIT quotient

Theorem

If G is reductive then $\mathbb{C}[X]^G$ is generated by finitely many homogeneous elements $q_0, \ldots, q_N$.

$X^{ss} = \left\{x \in X \mid \exists f \in \mathbb{C}[X]^G \text{ s.t. } f(x) \neq 0\right\}$. $X^{ss}//G = \mathrm{proj}(\mathbb{C}[X]^G)$. The map $\pi : X^{ss} \to X^{ss}//G$ given by $x \mapsto [q_0(x) : \cdots : q_N(x)]$ is called **Projective GIT quotient**.

Torus action

Let r, n are coprime.

1. T acts on $\mathbb{C}^{n\times r}$, and hence on $G_{r,n}$, by left multiplication.

2. Schubert cells are T-stable. T action on BwP is consumed in B, by writing $B = TU$. Hence Schubert varieties are T-stable.

3. Similarly Richardson varieties are T-stable. T action on $BwP \cap B^{-}vP$ is consumed in B, B^{-}, by writing $B = TU, B^{-} = TU^{-}$.

Aim: Understand $T\backslash\backslash G_{r,n}^{ss}$.

Known results

1. Gelfand-MacPherson correspondence.

2. $T\backslash\backslash G_{2,n}^{ss}$ is well understood.

3. Next one is $G_{3,n}$. Not much is know in this case. Simplest case is when $n = 7$. $T\backslash\backslash G_{3,7}^{ss}$ is projectively normal with line bundle $\mathcal{L}(7\omega_r)$.

   $\vdots$

Can we use Schubert varieties and Richardson varieties to understand the quotient?

Minimal Schubert variety

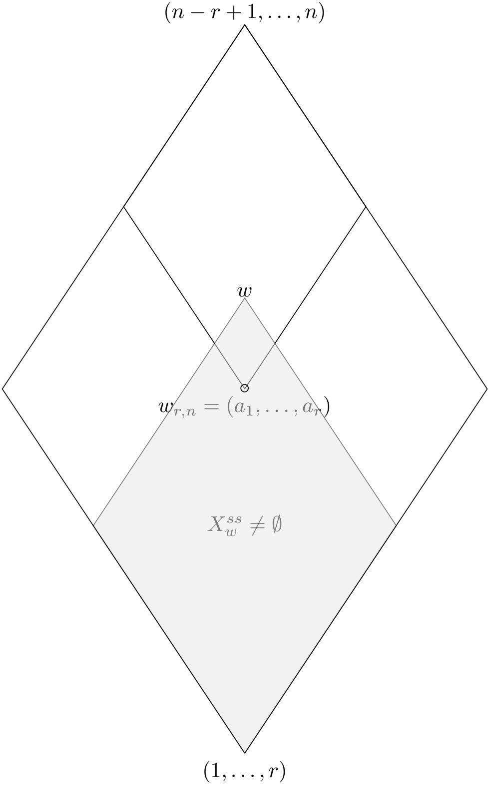
admitting semistable points.

From now on we fix the line bundle $\mathcal{L}(\omega_r)$ on Grassmannian.

Let $a_i = \lceil \frac{in}{r} \rceil, w_{r,n} = (a_1, \ldots, a_r)$.

Theorem: Kannan, Sardar 09

Let X_w be a Schubert variety. $X_w^{ss} \neq \emptyset$ if and only if $w_{r,n} \leq w$.



$(n-r+1,\ldots,n)$
w
$w_{r,n}=(a_1,\ldots,a_r)$
$X_w^{ss}\neq\emptyset$
$(1,\ldots,r)$

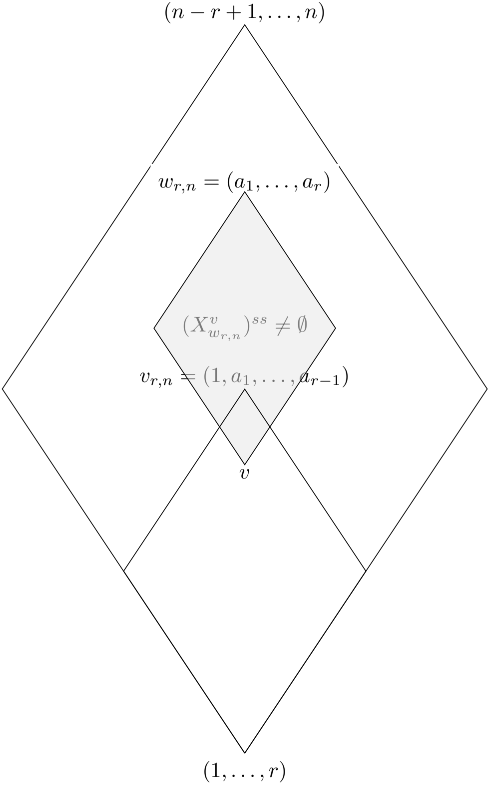
Richardson varieties in $X_{w_{r,n}}$

1. Even $T\backslash\backslash X^{ss}_{w_{r,n}}$ looks difficult. $T\backslash\backslash X^{ss}_{w_{3,7}}$ is understood.

2. Simplify the problem: Look at Richardson varieties inside $X_{w_{r,n}}$.

Theorem

Let $v_{r,n} = (1, a_1, \ldots, a_{r-1})$. $(X^{v}_{w_{r,n}})^{ss} \neq \emptyset$ if and only if $v \leq v_{r,n}$.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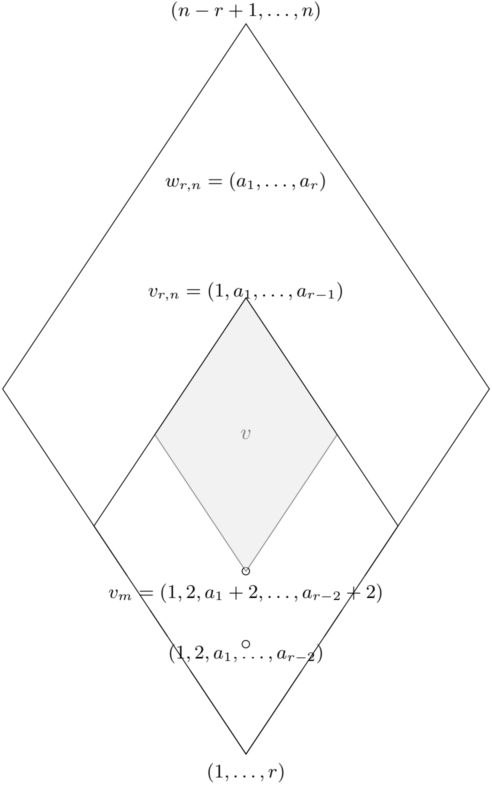


$(n-r+1,\ldots,n)$
$w_{r,n}=(a_1,\ldots,a_r)$
$v_{r,n}=(1,a_1,\ldots,a_{r-1})$
v
$v_m=(1,2,a_1+2,\ldots,a_{r-2}+2)$
$(1,2,a_1,\ldots,a_{r-2})$
$(1,\ldots,r)$

Main Theorem

Theorem

Let $c_i = ra_i - ni$. The GIT quotient $T\backslash\backslash X^v_{w_{r,n}}\mathcal{L}(n\omega_r)$ is isomorphic to $\mathbb{P}^{a_1-v(1)} \times \mathbb{P}^{a_2-v(2)} \times \cdots \times \mathbb{P}^{a_{r-1}-v(r-1)}$ and it is embedded via the very ample line bundle $\mathcal{O}_{\mathbb{P}^{a_1-v(1)}}(c_1) \boxtimes \mathcal{O}_{\mathbb{P}^{a_2-v(2)}}(c_2) \boxtimes \cdots \boxtimes \mathcal{O}_{\mathbb{P}^{a_{r-1}-v(r-1)}}(c_{r-1})$.

Plucker embedding

1. Define vector space $\wedge^r \mathbb{C}^n$ as $\mathbb{C}$ span of $\left\{ e_{\underline{i}} = e_{i_1} \wedge \cdots \wedge e_{i_r} \mid \underline{i} \in I_{r,n} \right\}$ and $\left\{ p_{\underline{i}} \mid \underline{i} \in I_{r,n} \right\}$ be the dual basis.

2. Define map $\phi : (\mathbb{C}^{n \times r})^o \to \mathbb{P}(\wedge^r \mathbb{C}^n)$ given by $(b_1, \ldots, b_r) \mapsto [b_1 \wedge \cdots \wedge b_r]$.

   We have $\mathrm{Im}(\phi) = G_{r,n}$.

3. We have compatible torus action on $\wedge^r \mathbb{C}^n$.

Tableau

Let $d\omega_r = ((d^r)) = (d, d, \dots, d)$ be a partition. We call Γ a column standard tableau of shape $d\omega_r$ if filling is from $\{1, \dots, n\}$, columns are strictly increasing. We call Γ semistandard if column entries are strictly increasing and row entries are weakly increasing. We allow columns to commute.

For each column standard tableau Γ we associate a monomial $f_\Gamma \in \mathbb{C}[\wedge^r \mathbb{C}^n]_d$. This is a bijection between monomials and column standard tableaux. Hence vector space basis of $\mathbb{C}[\wedge^r \mathbb{C}^n]$ is indexed by column standard tableaux. If Γ is semistandard then f_Γ is called standard monomial.

Ideal of X_w^v

Theorem

1. Let I be the ideal generated by Plucker relations. Then $\mathbb{C}[G_{r,n}] \simeq \frac{\mathbb{C}[\wedge^r \mathbb{C}^n]}{I}$.
2. Standard monomials form a vector space basis of $\mathbb{C}[G_{r,n}]$.
3. Ideal of Richardson variety X_w^v in $\mathbb{C}[\wedge^r \mathbb{C}^n]$ is generated by Plucker relations and $\{p_{\underline{i}} \mid \underline{i} \not\leq w \text{ or } \underline{i} \not\geq v\}$

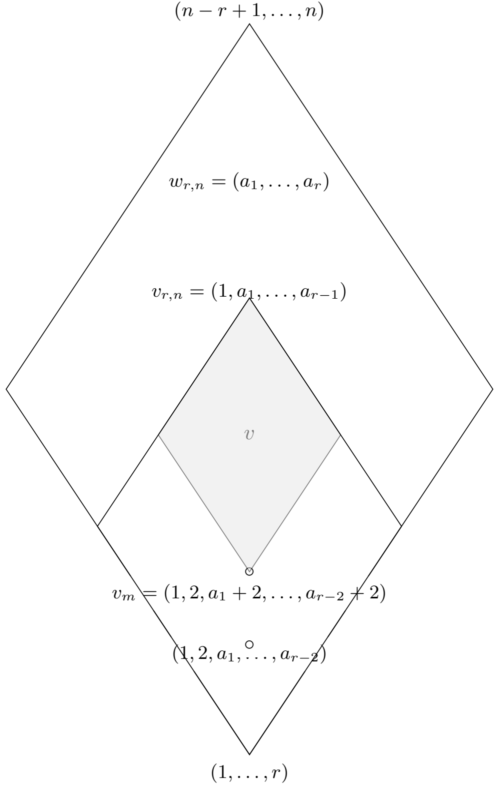
Richardson varieties of our interest

Recall

$$a_i = \lceil \tfrac{in}{r} \rceil, \qquad w_{r,n} = (a_1, \dots, a_r),$$
$$v_{r,n} = (1, a_1, \dots, a_{r-1}), \quad v_m = (1, 2, a_1 + 2, \dots, a_{r-2} + 2)$$

Let $v \in I_{r,n}$ be such that $v_m \leq v \leq v_{r,n}$. Richardson varieties of our interest are $X^v_{w_{r,n}}$.

Next we try to understand $\mathbb{C}[X^v_{w_{r,n}}]^T$.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$(1,2,a_1,\ldots,a_{r-2})$
$(1,\ldots,r)$

Ring of Invariants

Theorem

Let Γ be a tableau of shape $nd\omega_r$ such that

1. Γ is semistandard.
2. Γ contains each integer $1 \leq k \leq n$ with multiplicity rd.
3. $\Gamma(i, j) \in \{v(i), \ldots, a_i\}$ for $1 \leq i \leq r$ and for $1 \leq j \leq nd$.

Then $\mathbb{C}[X_w^v]^T$ is spanned by f_Γ.

Proof sketch

Let $Y = \mathbb{P}^{a_1-v(1)} \times \mathbb{P}^{a_2-v(2)} \times \cdots \times \mathbb{P}^{a_{r-1}-v(r-1)}$

and $\mathcal{M} = \mathcal{O}_{\mathbb{P}^{a_1-v(1)}}(c_1) \boxtimes \mathcal{O}_{\mathbb{P}^{a_2-v(2)}}(c_2) \boxtimes \cdots \boxtimes \mathcal{O}_{\mathbb{P}^{a_{r-1}-v(r-1)}}(c_{r-1})$.

Note that Y is a toric variety. Let S be a semigroup and $\mathbb{C}[S]$ be the semigroup algebra of embedding of Y embedded via $\mathcal{M}$.

We construct an isomorphism between graded algebra $\mathbb{C}[X^v_{w_{r,n}}]^T$ and $\mathbb{C}[S]$.

Proof sketch

bijection between graded basis.

Proof sketch

from bijection to ring map.