1 Quiz 1

Problem 1.1. *Prove that for any Lie algebra L over $\mathbb{F}$, $ad : L \to gl(L)$ is homomorphism of Lie algebras.*

Proof. Recall $ad : L \to gl(L)$ is defined by $x \mapsto (ad(x) : y \mapsto [x, y])$ for $x, y \in L$. Let $x, y, z \in L$ and $a, b, c \in \mathbb{F}$ be arbitrary elements.
1. Observe that $ad(x) \in gl(L)$ using right linearity of bracket:

$$ad(x)(by + cz) = [x, by + cz] = b[x, y] + c[x, z] = b \cdot ad(x)(y) + c \cdot ad(x)(z).$$

2. Next we prove ad is linear using left linearity of bracket:

$$ad(ax + by)(z) = [ax + by, z] = a[x, z] + b[y, z] = a \cdot ad(x)(z) + b \cdot ad(y)(z).$$

Hence we have $ad(ax + by) = a \cdot ad(x) + b \cdot ad(y)$.
3. Lastly we want to show that ad preserves bracket:

$$\begin{aligned} ad([x, y])(z) &= [[x, y], z] = -[[y, z], x] - [[z, x], y] = [x, [y, z]] - [y, [x, z]] \\ &= ad(x)\big(ad(y)(z)\big) - ad(y)\big(ad(x)(z)\big) \\ &= \big(ad(x) \circ ad(y) - ad(y) \circ ad(x)\big)(z) \\ \therefore ad([x, y]) &= [ad(x), ad(y)]. \end{aligned}$$

□

Problem 1.2. *Give an example of Lie algebra L such that $ad(L)$ is nonzero Lie subalgebra of $sl(L)$.*

Examples. 1. Let $L = sl(2, \mathbb{C})$. Since $sl(2, \mathbb{C})$ is simple and not abelian, we have $[L, L] = L$, hence $gl(L) \supset ad(L) = ad([L, L]) = [ad(L), ad(L)]$. This implies for any $x \in ad(L)$ there are $y, z \in ad(L)$ such that $x = [y, z]$. Hence $trace(x) = trace([y, z]) = 0$. We have $ad(L) \subset sl(L)$. $ad(L) \neq 0$ follows from L is not abelian.
Alternate proof for $L = sl(2, \mathbb{C})$. Basis of L is $e_{11} - e_{22}, e_{12}, e_{21}$ in that order. Observe that

$$ad(e_{11} - e_{22}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix}, ad(e_{12}) = \begin{pmatrix} 0 & 0 & 1 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, ad(e_{21}) = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}$$

Hence $ad(L) \subset sl(L)$.
2. For any simple Lie algebra L over char 0 field. Proof is similar to that of above.
3. Let L be nilpotent lie algebra then every element $x \in ad(L)$ is ad-nilpotent. This implies all eigen values of $ad(x)$ are 0. Understand all steps in between.
4. Let $L = \mathbb{R}^3$ with bracket defined by cross product. First prove that this is indeed Lie algebra by verifying all axioms. Second write matrices of $ad(x)$ for all basis elements x, and obeserve that trace of each of them is 0. □